Extended Object Filtering using Spatial Independent Cluster Processes

Anthony Swain & **Daniel Clark**
Joint Research Institute in Signal and Image Processing
Heriot-Watt University
Edinburgh, UK.
ajs27@hw.ac.uk & d.e.clark@hw.ac.uk

Abstract – *Recent research into multi-object filtering for non-standard targets introduced alternative approaches for target group representation. In these approaches a measurement model (likelihood) was suggested that led to a representation of the measurements as a spatial point process, namely a Poisson point process. In this paper we take a more traditional approach to extended target tracking. We assume a* 'standard' *measurement model (at most one measurement generated from a target point), but represent the target group (*extended targets*) as a spatial cluster process, in particular an independent cluster process with a fixed distribution on the component (daughter) process. With this assumption we are able to derive approximate measurement-update equations for the first order moment density of the extended object Bayes filter in a number of scenarios. Such approximations are Bayes optimal and provide estimates for the number of clusters (*extended targets*) and their locations.*

Keywords: Tracking, filtering, estimation, spatial cluster processes.

1 Introduction

There have been several approaches proposed in recent years for modelling extended target tracking problems which include the use of random matrices to represent the physical extension [1, 2] and modelling extended objects as circular discs in two dimensions [3]. Amongst these, of particular interest for the scope of this paper, is the approach which proposes a measurement model (likelihood) that leads to measurement representations over the sensor observation region as a spatial point process, in particular, a Poisson point process. The problem for which this model addresses is one which describes a scenario where a target is close to a sensor, resulting in possible back-reflections. When back-reflections occur, the result is an image of the target consisting of a large number of point detections. Targets that generate measurement-sets of this kind are otherwise known as *extended targets*. This was first proposed by Gilholm, Godsill, Maskell, and Salmond [4], where a particle filter implementation was suggested, and since developed by Mahler [5], deriving the measurement-update equation for a *probability hypothesis density* (PHD) filter, which approximates the first-order moment density of the multi-target Bayes filter.

In the PHD filter for this non-standard multi-target measurement model [5], an assumption was imposed on the predicted multi-target distribution in the measurement-update of the Bayes filter, that it is approximately Poisson. This was later generalized further by Mahler [6] by proposing a Poisson mixture process to represent the measurement model and imposing the assumption on the predicted multi-target distribution that it is i.i.d. (independent, identically distributed).

In this paper an alternative approach to extended target tracking problems is adopted. Rather than using the measurement model proposed by Gilholm *et al.* [4] we assume that at most a single measurement is generated by a point target, but that these point targets belong to groups or *clusters*. For this we use a concept from point process theory of spatial cluster processes to represent a group-target state-set, the details for which we provide in the Section 1.1. However we derive an approximate measurement-update equation, similar to Mahler [5, 6], for the first-order moment density of the multi-object Bayes filter, imposing an assumption on the predicted multi-group multi-target distribution.

1.1 Spatial cluster processes

Cluster processes are an essential concept in the theory of point process [7], stochastic geometry [8, 9], modelling spatial population processes [10], and random finite sets [11]. Examples of their applications in statistics include clustering of galaxies [12], rainfall modelling [13] and epidemiology [14].

Cluster processes are described as a superposition of point processes of cluster centres, an unseen point process, to which are associated a random number of points forming component processes, occupying an observable metric space. We refer to the cluster centre processes as *parent processes* and their component processes as *daughter processes*. This is illustrated in Figure 1 with a simulated example of a Matern cluster process. In general, cluster processes can be characterised by their *probability generating functionals* (p.g.fl). Probability generating functionals are an important concept in point process theory which uniquely characterises probability distributions. So before we state the general characterisation for cluster processes, it is necessary to define the notion of a p.g.fl.

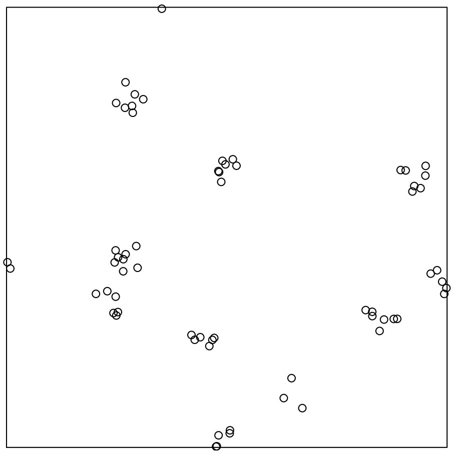

Figure 1: Realisation of a Matern cluster process with parent Poisson rate 10, and daughter process Poisson rate 6 with radius 0.04

Definition 1 (Probability Generating Functionals). *The p.g.fl of a multi-object probability density $f(X)$ for a set of points $X = \{x_1, \dots, x_n\}$ is*

$$G[h] := E(h^X) = \int h^X\, f(X)\, \delta X, \tag{1}$$

where for any "test function" $h(x)$ with $0 \le h(x) \le 1$ we have

$$h^X = \begin{cases} 1 & \text{for } X = \emptyset\ , \\ \prod_{x\in X} h(x) & \text{for } X = \{x_1, \dots, x_n\}\ . \end{cases} \tag{2}$$

For the integral in (1) we adopt the concept of the set integral from *Finite Set Statistics* (FISST) such that,

$$\int f(X)\, \delta X := f(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int f(\{x_1, \dots, x_n\})\, dx_1 \cdots dx_n. \tag{3}$$

Probability generating functionals are a useful mathematical tool for deriving moments of multi-object probability densities, as proposed by Moyal [10]. For a multi-object probability density $f(X)$, the nth-order moment density can be obtained by taking the functional derivative of a p.g.fl $G[h]$, given by (1), with respect to $X = \{x_1, \dots, x_n\}$ and evaluating at $h = 1$. That is, using the shorthand notation for functional derivatives from FISST [15], the nth-order moment density of $f(X)$ is

$$D(X) = \frac{\delta G}{\delta X}[h]\bigg|_{h=1} = \frac{\delta G}{\delta x_1 \cdots \delta x_n}[h]\bigg|_{h=1} \tag{4}$$

and in particular the first order moment density, otherwise known in target tracking literature ([15]) as the *Probability Hypothesis Density*, is

$$D(x) = \frac{\delta G}{\delta x}[h]\bigg|_{h=1} \tag{5}$$

To illustrate the use of probability generating functionals for the derivation of first-order moments, we now provide an example for an i.i.d. cluster.

Example 1 (Independent, identically distributed (i.i.d) cluster). *Let $p(n)$ be a probability distribution on the non-negative integers and let $s(y)$ be a probability density function. For any $Y = \{y_1, \dots, y_n\}$ with $|Y| = n$, the probability density given by*

$$f(Y) = n!\, p(n) \prod_{i=1}^{n} s(y_i) \tag{6}$$

is known as an i.i.d. cluster distribution. The probability distribution $p(n) = Pr(|Y| = n)$ is referred to as the cardinality distribution.

Now the p.g.fl of an i.i.d cluster distribution satisfies

$$G[h] = G(s[h]) \tag{7}$$

where $s[h] = \int s(y)h(y)\, dy$ and $G(y) = \sum_{n=0}^{\infty} p(n)y^n$ is the probability generating function of the cardinality distribution, which in turn can be recovered from the p.g.f. as follows

$$p(n) = \frac{1}{n!} G^{(n)}(0). \tag{8}$$

where $G^{(n)}$ is the nth derivative of the p.g.f. The intensity of an i.i.d. cluster, or its first-order moment density, is found by taking the functional derivative of its p.g.fl with respect to h and evaluating at $h = 1$,

$$D(y) = \frac{\delta G}{\delta y}[h]\bigg|_{h=1} = \frac{\delta G}{\delta y}(s[h])\bigg|_{h=1} = G^{(1)}(1)\, s(y) \tag{9}$$

1.1.1 General Cluster Process

General cluster processes are characterised by a component (daughter) process p.g.fl, G_x, within a parent cluster centre p.g.fl, G_c,

$$G_c\big[G_x[h \mid \cdot]\big], \tag{10}$$

where $G_x[h \mid \cdot]$, which is treated as an argument of G_c, is the p.g.fl of the daughter process for any particular realisation of the centre (parent) process.

In this paper we shall consider the independent cluster process for the purpose of approximating a generalised measurement-update equation. The independent cluster process is a doubly-stochastic process in which the parent and daughter processes are i.i.d. clusters.

It is convenient to model the extended objects as an independent cluster process, since the cardinality and spatial distribution in the daughter process is completely determined by the state position of its parent in relation to the sensor.

1.2 Extended object Bayes filtering

We now define the recursive equations of the Bayes filter for an extended object state system.

Let $Z^{(k)}: Z_1, \dots, Z_k$ be a time-sequence of multi-object measurement sets at times $1, \dots, k$. Each measurement set $Z_l = \{z_{1,l}, \dots, z_{m,l}\}$ at time-step l consists of a random finite set of measurements, where each measurement is in some observation space. Suppose at time k we have an extended object state set $\mathbb{X}_k = \{(c_1, X_1), \dots, (c_n, X_n)\}$, where we define $c_1, \dots, c_n$ as the cluster centres (or the true targets) and corresponding extended target sets $X_i = \{x_{i,1}, \dots, x_{i,N(i)}\}$ for each cluster centre c_i.

Denote $\mathbb{X} = \mathbb{X}_{k+1}$ and $\mathbb{X}' = \mathbb{X}_k$ and suppose that $f_{k|k}(\mathbb{X}' \mid Z^{(k)})$ is known. Then the recursive equations for the predicted and updated extended-object posterior densities are

$$f_{k+1|k}(\mathbb{X} \mid Z^{(k)}) = \int f_{k+1|k}(\mathbb{X} \mid \mathbb{X}')\, f_{k|k}(\mathbb{X}' \mid Z^{(k)})\, \delta\mathbb{X}' \tag{11}$$

$$f_{k+1|k+1}(\mathbb{X} \mid Z^{(k+1)}) = \frac{f_{k+1}(Z_{k+1} \mid \mathbb{X})\, f_{k+1|k}(\mathbb{X} \mid Z^{(k)})}{\int f_{k+1}(Z_{k+1} \mid \mathbb{X})\, f_{k+1|k}(\mathbb{X} \mid Z^{(k)})\, \delta\mathbb{X}} \tag{12}$$

where $f_{k+1|k}(\mathbb{X} \mid \mathbb{X}')$ is a group-target Markov transition density and $f_{k+1}(Z_{k+1} \mid \mathbb{X})$ is a group-target likelihood. The prediction formula in (11) corresponds to the Chapman-Kolmogov equation and the measurement-update formula corresponds to Bayes rule. They are explicitly modelled in the FISST framework where the integrals in (11) & (12) are generalised group-target set integrals defined by

$$\int f(\mathbb{X})\, \delta\mathbb{X} = \sum_{n=0}^{\infty} \frac{1}{n!} \iint f\left(\{(c_1, X_1), \dots, (c_n, X_n)\}\right) dc_1 \cdots dc_n \delta X_1 \cdots \delta X_n \tag{13}$$

The Bayes filter described here cannot be implemented in a computationally tractable manner which means an suitable approximation is required. Among the approximations that were recently proposed for multi-target tracking problems of particular importance are the PHD and CPHD filters [16, 17]. Instead of propagating the full multi-target posterior density $f_{k|k}(X_k \mid Z^{(k)})$, the PHD filter propagates its first order moment, referred to as the Probability Hypothesis Density (PHD) or intensity function. The Cardinalized Probability Hypothesis Density (CPHD) filter also propagates the cardinality distribution. The recursive equations for the first-moment densities and cardinality distributions in the PHD and CPHD filters are found with the use of probability generating functionals (p.g.fl.) and probability generating functions as described in Definition 1 and Example 1.

In the next section we present the main results for the first-order moment densities of the recursion in the extended object Bayes filter where we present a number of cases for the measurement-update. Details of the derivations for the measurement-update results are provided in Section 3.

2 Extended object first-moment filter

2.1 Prediction

Before stating the first-moment density formula in the prediction step of the extended object Bayes filter we first note that, since the extended objects are being modelled as an independent cluster process and due to the static nature of the daughter process within this cluster process, we have a 'standard' motion model which describes only the dynamics on the true targets (cluster centres). This motion is governed by a Markov transition density $f_{k+1|k}(c \mid c')$ with a probability of survival $p_S(c')$.

Theorem 1. *The first-moment density of the prediction equation at time-step $k+1$ in the extended object Bayes filter is*

$$\gamma_{k+1|k}(c) + \int D_{k|k}(c')\, p_S(c')\, f_{k+1|k}(c \mid c')\, dc' \tag{14}$$

where

$\gamma_{k+1|k}(c) =$ *the first-moment of the group-target density, which describes the emergence of new group-targets at time-step $k+1$,*

$D_{k|k}(c') =$ *the first-moment of the group-target posterior density at time-step k,*

$p_S(c') =$ *the probability that a group-target state c' survives at time-step $k+1$,*

$f_{k+1|k}(c \mid c') =$ *the Markov transition density of group-target state c at time-step $k+1$ given group-target state c' at time-step k.*

2.2 Measurement-update

The p.g.fl of the updated posterior density, given by (12) in the extended object Bayes filter, is

$$\begin{aligned} G_{k+1|k+1}[h] &= \int h^{\mathbb{X}} f_{k+1|k+1}(\mathbb{X} \mid Z^{(k+1)})\, \delta\mathbb{X} \\ &= \int \frac{h^{\mathbb{X}} f_{k+1}(Z_{k+1} \mid \mathbb{X}) f_{k+1|k}(\mathbb{X} \mid Z^{(k)})}{\int f_{k+1}(Z_{k+1} \mid \mathbb{Y}) f_{k+1|k}(\mathbb{Y} \mid Z^{(k)})\, \delta\mathbb{Y}}\, \delta\mathbb{X}\,. \end{aligned} \tag{15}$$

For convenience, we denote $Z = Z_{k+1}$ and define a two variable p.g.fl $F[g,h]$ as follows

$$\begin{aligned} F[g,h] &= \int \int h^{\mathbb{X}}\, g^Z\, f_{k+1}(Z \mid \mathbb{X})\, f_{k+1|k}(\mathbb{X} \mid Z^{(k)})\, \delta\mathbb{X}\, \delta Z \\ &= \int h^{\mathbb{X}} \underbrace{\left(\int g^Z\, f_{k+1}(Z \mid \mathbb{X})\, \delta Z \right)}_{G_{k+1}[g|\mathbb{X}]} f_{k+1|k}(\mathbb{X} \mid Z^{(k)})\, \delta\mathbb{X}\,. \end{aligned} \tag{16}$$

The p.g.fl $G_{k+1|k+1}[h]$ can be derived from the functional derivatives of $F[g,h]$

$$G_{k+1|k+1}[h] = \left. \frac{\delta F}{\delta Z_{k+1}}[g,h] \right|_{g=0} \Big/ \left. \frac{\delta F}{\delta Z_{k+1}}[g,h] \right|_{g=0,h=1}, \tag{17}$$

a result which was proved by Mahler [16].

In order to formulate the two variable p.g.fl. $F[g,h]$, it is necessary to specify an observation model. For this we use the 'standard' multi-target measurement model, where a single sensor collects measurements from multiple possible targets such that no target generates more than one measurement and vice versa. In this model the measurement generating targets in question correspond to the extended targets in the daughter process, which are conditional on the true target (cluster centre), of the cluster process.

Theorem 2 (Base Case - No false alarms or missed detections)**.** *The first moment density of the measurement-update equation at time-step $k+1$ in the extended object Bayes filter for this case is*

$$s_1(c)\sum_{P\in\mathcal{P}(Z)}\omega_P\left(\eta_P+\sum_{W\in P}\frac{L_W(c)}{s_1[L_W]}\right) \tag{18}$$

where

$$\begin{aligned}
\omega_P &= \frac{G_c^{(|P|)}(\,.\,)\prod_{W\in P}s_1[L_W]}{\displaystyle\sum_{Q\in\mathcal{P}(Z)}G_c^{(|Q|)}(\,.\,)\prod_{W\in Q}s_1[L_W]},\\
\eta_P &= \frac{G_c^{(|P|+1)}(\,.\,)\cdot p_x(0\mid c)}{G_c^{(|P|)}(\,.\,)},\\
G_c^{(n)}(\,.\,) &= G_c^{(n)}\Big(s_1\big[p_x(0\mid\cdot)\big]\Big),\\
L_W(c) &= |W|!\,p_x(|W|\mid c)\prod_{z\in W}s_2[L_z\mid c].
\end{aligned}$$

$L_z = L_z(x\mid c) =$ $f_{k+1}(z\mid x,c)$ = *single-object likelihood for individual measurement $z\in Z_{k+1}$*

$s_1(c) =$ $s_{k+1|k}(c)$ = *predicted probability density function of the parent process in the independent cluster process at time-step $k+1$*

$s_2 = s_{k+1|k} =$ *predicted probability density function of the daughter process at time-step $k+1$*

$G_c^{(n)} =$ *the nth derivative of the p.g.fl of the predicted cardinality, p_c, for the parent process at time-step $k+1$*

$p_x =$ *predicted cardinality of the daughter process at time-step $k+1$*

Remark 1. *The first summation in* (18) *is taken over all partitions of the measurement set Z, where a partition, P, is defined such that the union of all its elements (cells W) is Z, i.e. $\cup_{W\in P}W = Z$.*

Corollary 1. *Suppose the daughter process in the predicted independent cluster process is Poisson such that it has p.g.fl $G_x[h] = G_x(s_2[h]) = \mathrm{e}^{\mathring{x}s_2[h]-\mathring{x}}$ with constant expected number of components, $\mathring{x}$, of the extended target and a uniform spatial distribution $s_2(x\mid c)$. The first-order moment density for the measurement-update in the case with no false alarms or missed detections is then*

$$s_1(c)\,\mathrm{e}^{-\mathring{x}}\sum_{P\in\mathcal{P}(Z)}\omega_P\left(\eta_P+\sum_{W\in P}\frac{\mathring{x}^{|W|}\prod_{z\in W}s_2[L_z\mid c]}{s_1\Big[\mathrm{e}^{-\mathring{x}}\prod_{z\in W}\mathring{x}s_2[L_z\mid c]\Big]}\right) \tag{19}$$

where

$$\begin{aligned}
\omega_P &= \frac{G_c^{(|P|)}\big(s_1[\mathrm{e}^{-\mathring{x}}]\big)\prod_{W\in P}s_1\left[\mathrm{e}^{-\mathring{x}}\prod_{z\in W}\mathring{x}s_2[L_z\mid c]\right]}{\displaystyle\sum_{Q\in\mathcal{P}(Z)}G_c^{(|Q|)}\big(s_1[\mathrm{e}^{-\mathring{x}}]\big)\prod_{W\in Q}s_1\Big[\mathrm{e}^{-\mathring{x}}\prod_{z\in W}\mathring{x}s_2[L_z\mid c]\Big]}\\
\eta_P &= \frac{G_c^{(|P|+1)}\big(s_1[\mathrm{e}^{-\mathring{x}}]\big)}{G_c^{(|P|)}\big(s_1[\mathrm{e}^{-\mathring{x}}]\big)}
\end{aligned}$$

The first-order moment density for the measurement-update equation in Corollary 1 is equivalent to a result by Mahler [6] in which he specifies a Poisson mixture process for the likelihood model and assumes the predicted posterior distribution is an i.i.d. cluster. The term $\prod_{z\in W}s_2[L_z\mid c]$ in (19) corresponds to the product of spatial measurement distributions $\theta(z\mid x)$ in the Poisson mixture model.

Theorem 3 (General Case - Independent cluster false alarms and missed detections)**.** *The first moment density of the measurement-update equation at time-step $k+1$ in the extended object Bayes filter in this case is*

$$\begin{aligned}
s_1(c)\Bigg\{&\sum_{P\in\mathcal{P}(Z)}\left(\omega_P\,\eta+\sum_{Q\subseteq P}\omega_Q\,\eta_Q\right)\Big(q_D(c)+p_D(c)\cdot p_x(0\mid c)\Big)\\
&+\sum_{P\in\mathcal{P}(Z)}\left(\sum_{Q\subseteq P}\omega_Q\sum_{W_q\in Q}\frac{L_{W_q}(c)}{s_1[p_D\cdot L_{W_q}]}\right)p_D(c)\Bigg\}
\end{aligned} \tag{20}$$

where

$$\omega_P = \frac{G_c(\,.\,)\,C_1^{(|P|)}(0)\prod_{W\in P}c_1[\,.\,]_W}{\displaystyle\sum_{P'}\sum_{Q\uplus R=P'}G_c^{(|Q|)}(\,.\,)\prod_{W_q\in Q}s_1[\,.\,]_{W_q}\,C_1^{(|R|)}(0)\prod_{W_r\in R}c_1[\,.\,]_{W_r}},$$

$$\omega_Q = \frac{G_c^{(|Q|)}(\,.\,)\prod_{W_q\in Q}s_1[\,.\,]_{W_q}\,C_1^{(|P|-|Q|)}(0)\prod_{\bar{W}_q\in P\setminus Q}c_1[\,.\,]_{\bar{W}_q}}{\displaystyle\sum_{P'}\sum_{Q\uplus R=P'}G_c^{(|Q|)}(\,.\,)\prod_{W_q\in Q}s_1[\,.\,]_{W_q}\,C_1^{(|R|)}(0)\prod_{W_r\in R}c_1[\,.\,]_{W_r}},$$

and

$$\begin{aligned}
\eta &= \frac{G_c^{(1)}(\,.\,)}{G_c(\,.\,)},\quad \eta_Q=\frac{G_c^{(|Q|+1)}(\,.\,)}{G_c^{(|Q|)}(\,.\,)},\\
G_c^{(n)}(\,.\,) &= G_c^{(n)}\Big(s_1\big[q_D+p_D\cdot p_x(0\mid\cdot)\big]\Big),\\
L_{W_q}(c) &= |W_q|!\,p_x(|W_q|\mid c)\prod_{z\in W_q}s_2[L_z\mid c],\\
s_1[\,.\,]_{W_q} &= s_1[p_D\cdot L_{W_q}],\\
c_1[\,.\,]_{\bar{W}_q} &= c_1\left[C_2^{(|\bar{W}_q|)}(0\mid\cdot)\prod_{w\in\bar{W}_q}c_2(w\mid\cdot)\right].
\end{aligned}$$

The following notation L_z, s_1, s_2, $G_c^{(n)}$ & p_x are defined as in Theorem 2 and in addition we denote

$p_D(c) =$ *the probability of detection where $q_D(c) = 1 - p_D(c)$,*

$C_1^{(i)}, C_2^{(j)} =$ *ith and jth derivatives of the p.g.fl of the cardinalities for the parent and daughter processes respectively in the independent cluster false alarm process,*

$c_1, c_2 =$ *probability density functions of the false alarm parent and daughter processes respectively.*

Remark 2. *The summation $\sum_{Q\uplus R=P'}$ is taken over all subpartitions $Q,\ R \subseteq P'$ of the measurement set Z, such that $Q \uplus R = P'$.*

Given the, admittedly contrived, scenario that the daughter spatial distribution is a delta function $\delta_x(c)$ and the daughter cardinality is as follows: $p_x(1 \mid c) = 1$, an i.i.d. cluster distribution can be obtained from an independent cluster process. Subsequently, the result in Theorem 3 immediately leads to the following measurement-update equation.

Corollary 2. *Suppose that, in the predicted independent cluster process, the daughter process is Poisson such that it has p.g.fl. $G_x[h] = G_x(s_2[h]) = \mathrm{e}^{\gamma(c)s_2[h]-\gamma(c)}$ with variable expected number of components, $\gamma(c)$, of the extended targets and the parent process is also Poisson such that its p.g.fl is $G_c[h] = G_c(s_1[h]) = \mathrm{e}^{\mu s_1[h]-\mu}$. Then the first-order moment density for the measurement-update in the case with missed detections and Poisson false alarms, i.e. $C_1[g] = C_1(c_1[g]) = \mathrm{e}^{\lambda c_1[g]-\lambda}$, is*

$$\mu\, s_1(c)\Bigg\{1 - p_D(c) + \mathrm{e}^{-\gamma(c)}p_D(c)\Bigg(1 + \sum_{P\in\mathcal{P}(Z)} \omega_P \sum_{W\in P} \frac{\gamma(c)^{|W|}}{\hat{s}_W} \prod_{z\in W} \frac{s_2[L_z \mid c]}{\lambda\, c_1(z)}\Bigg)\Bigg\} \tag{21}$$

where, if $\delta_{|W|,1}$ is the Kronecker delta,

$$\omega_P = \left(\prod_{W\in P} \hat{s}_W\right)\Bigg/ \sum_{Q\in\mathcal{P}(Z)} \left(\prod_{W\in Q} \hat{s}_W\right)$$

$$\hat{s}_W = \delta_{|W|,1} + \mu\, s_1\left[\mathrm{e}^{-\gamma}\gamma^{|W|}p_D \prod_{z\in W} \frac{s_2[L_z \mid c]}{\lambda\, c_1(z)}\right]$$

The result in Corollary 2 is equivalent to Mahler's Extended Targets PHD filter measurement-update equation [5], in which he assumes a generalised Poisson model for the single-target likelihood function due to Gilholm *et al.* [4]. The term $\prod_{z\in W} s_2[L_z \mid c]$ in (21) corresponds to the spatial measurement distribution $\phi_z(x)$ in the generalised Poisson model.

3 Mathematical proofs

3.1 Proof of Theorem 2 & 3

Under the assumption that the predicted probability distribution $f_{k+1|k}(\mathbb{X} \mid Z^{(k)})$ is described by an independent cluster process, the two variable p.g.fl, $F[g,h]$ in the *General Case*, where independent cluster false alarms and missed detections are included in the model, can be written as

$$F[g,h] = G_c\big[h \cdot G_{p_D}[G_x[G_L[g \mid \cdot] \mid \cdot]]\big] \cdot C_1\big[C_2[g \mid \cdot]\big] \tag{22}$$

where

$$\begin{aligned} G_c[h] &= \sum_{n=0}^{\infty} p_c(n)\left(\int s_1(u)h(u)\,du\right)^n \\ &= \text{p.g.fl for the parent process in the group-target independent cluster process,} \end{aligned}$$

$$\begin{aligned} G_x[h \mid u] &= \sum_{m=0}^{\infty} p_x(m \mid u)\left(\int s_2(w \mid u)h(w)\,dw\right)^m \\ &= \text{p.g.fl for the group-target daughter process,} \end{aligned}$$

$$\begin{aligned} G_{p_D}[h \mid u] &= q_D(u) + p_D(u)h(u) \\ &= \text{the process of thinning or random deletion where } p_D(u) \text{ is the probability that an object } u \text{ is detected and } q_D(u) = 1 - p_D(u), \end{aligned}$$

$$\begin{aligned} G_L[g \mid \cdot] &= \int g(z)L_z(\cdot)\,dz \\ &= \text{conditional likelihood functional,} \end{aligned}$$

$$\begin{aligned} C_1[g] &= \sum_{n=0}^{\infty} p_{\kappa_1}(n)\left(\int c_1(r)g(r)\,dr\right)^n \\ &= \text{p.g.fl for the parent process in the independent cluster false alarm process,} \end{aligned}$$

$$\begin{aligned} C_2[g \mid r] &= \sum_{m=0}^{\infty} p_{\kappa_2}(m \mid r)\left(\int c_2(z \mid r)g(z)\,dz\right)^m \\ &= \text{p.g.fl for the false alarm daughter process,} \end{aligned}$$

For the derivation of the *intensity function* in the General Case, we consider the functional derivative with respect to g with respect to Z_{k+1} of $F[g,h]$ from the result for the p.g.fl $G_{k+1|k+1}[h]$ in (17) and prove the following Lemma.

Lemma 1. *The functional derivative of the two variable p.g.fl $F[g,h]$ given in (22) with respect to g with respect $Z = Z_{k+1}$ is*

$$\begin{aligned} \frac{\delta F}{\delta Z}[g,h] = \sum_{P\in\mathcal{P}(Z)} \sum_{Q\uplus R=P} G_c^{(|Q|)}(\,.\,) \prod_{W_q\in Q} s_1[\,.\,]_{W_q} \\ \times C_1^{(|R|)}\Big(c_1\big[C_2(c_2[g \mid \cdot])\big]\Big) \prod_{W_r\in R} c_1[\,.\,]_{W_r} \end{aligned} \tag{23}$$

where

$$\begin{aligned} G_c^{(i)}(\,.\,) &= G_c^{(i)}\Big(s_1\big[h(q_D + p_D\, G_x(s_2[G_L[g \mid \cdot] \mid \cdot]))\big]\Big), \\ s_1[\,.\,]_{W_q} &= s_1\left[h\, p_D\, G_x^{(|W_q|)}(s_2[G_L[g \mid \cdot] \mid \cdot]) \prod_{z\in W_q} s_2[L_z \mid \cdot]\right], \\ c_1[\,.\,]_{W_r} &= c_1\left[C_2^{(|W_r|)}(c_2[g \mid \cdot]) \prod_{w\in W_r} c_2(w \mid \cdot)\right]. \end{aligned} \tag{24}$$

Proof. To prove the result in Lemma 1 we use mathematical induction on the number, $|Z| = m$, of elements of the measurement set Z. For the initial step of the induction, assume $|Z| = 1$ and that $Z = \{z_1\}$. In this case, there is only one partition: $P = \{z_1\}$. Now taking the functional derivative of $F[g.h]$ with respect to g with respect to $Z = \{z_1\}$, where $F[g,h]$ is given by (22), we have

$$\begin{aligned} \frac{\delta F}{\delta z_1}[g,h] &= G_c(\,.\,)\; C_1^{(1)}(\,.\,)\; c_1\Big[C_2^{(1)}(c_2[g \mid \cdot]) \cdot c_2(z_1\cdot)\Big] \\ &+\; G_c^{(1)}(\,.\,)\; s_1\Big[h\, p_D\, G_x^{(1)}(s_2[G_L[g \mid \cdot]]) \cdot s_2[L_{z_1} \mid \cdot]\Big]\; C_1(\,.\,) \end{aligned} \tag{25}$$

where we denote the ith and jth derivative of G_c and C_1 respectively by

$$\begin{aligned} G_c^{(i)}(\,.\,) = &\ G_c^{(i)}\Big(s_1\big[h(q_D + p_D \cdot G_x(s_2[G_L[g \mid \cdot] \mid \cdot]))\big]\Big), \\ C_1^{(j)}(\,.\,) = &\ C_1^{(j)}\Big(c_1\big[C_2(c_2[g \mid \cdot])\big]\Big), \end{aligned}$$

which for $i = 0$ and $j = 0$ gives $G_c^{(i)}(\,.\,) = G_c(\,.\,)$ and $C_1^{(j)}(\,.\,) = C_1(\,.\,)$. The result in (25) is as claimed in (23) with $P = \{z_1\}$ for $|Z| = 1$ and thus the result holds for the initial step.

Now for the next part of the inductive proof, assume the result holds for $|Z| = m$, where $Z = \{z_1, \ldots, z_m\}$ and take the functional derivative of $F[g,h]$ with respect to g with respect to $Z \cup \{z_{m+1}\}$, which gives

$$\begin{aligned} &\frac{\delta^2 F}{\delta Z\, \delta z_{m+1}}[g,h] = \\ &\sum_{P\in\mathcal{P}(Z)} \sum_{Q\uplus R=P} \left(\frac{\delta}{\delta z_{m+1}} G_c^{(|Q|)}(\,.\,)\right) \prod_{W_q\in Q} s_1[\,.\,]_{W_q} \\ &\qquad \times C_1^{(|R|)}(\,.\,) \prod_{W_r\in R} c_1[\,.\,]_{W_r} \\ &+ \sum_{P\in\mathcal{P}(Z)} \sum_{Q\uplus R=P} G_c^{(|Q|)}(\,.\,)\, \frac{\delta}{\delta z_{m+1}} \left(\prod_{W_q\in Q} s_1[\,.\,]_{W_q}\right) \\ &\qquad \times C_1^{(|R|)}(\,.\,) \prod_{W_r\in R} c_1[\,.\,]_{W_r} \\ &+ \sum_{P\in\mathcal{P}(Z)} \sum_{Q\uplus R=P} G_c^{(|Q|)}(\,.\,) \prod_{W_q\in Q} s_1[\,.\,]_{W_q} \\ &\qquad \times \left(\frac{\delta}{\delta z_{m+1}} C_1^{(|R|)}(\,.\,)\right) \prod_{W_r\in R} c_1[\,.\,]_{W_r} \\ &+ \sum_{P\in\mathcal{P}(Z)} \sum_{Q\uplus R=P} G_c^{(|Q|)}(\,.\,) \prod_{W_q\in Q} s_1[\,.\,]_{W_q} \\ &\qquad \times C_1^{(|R|)}(\,.\,)\, \frac{\delta}{\delta z_{m+1}} \left(\prod_{W_r\in R} c_1[\,.\,]_{W_r}\right) \end{aligned}$$

The functional derivatives of $G_c^{(|Q|)}(\,.\,)$ and $\prod_{W_q\in Q} s_1[\,.\,]_{W_q}$ with respect to z_{m+1} are found to be

$$\begin{aligned} &\frac{\delta}{\delta z_{m+1}} G_c^{(|Q|)}(\,.\,) \\ &= G_c^{(|Q|+1)}(\,.\,)\ s_1\Big[h\, p_D\, G_x^{(1)}\big(s_2[G_L[g \mid \cdot] \mid \cdot]\big) \cdot s_2[L_{z_{m+1}} \mid \cdot]\Big] \\ &= G_c^{(|Q|+1)}(\,.\,)\ s_1[\,.\,]_{\{z_{m+1}\}}, \end{aligned}$$

and

$$\begin{aligned} &\frac{\delta}{\delta z_{m+1}} \left(\prod_{W_q\in Q} s_1[\,.\,]_{W_q}\right) \\ &= \left(\prod_{W_q\in Q} s_1[\,.\,]_{W_q}\right) \sum_{W_q\in Q} \frac{1}{s_1[\,.\,]_{W_q}}\ \frac{\delta}{\delta z_{m+1}} s_1[\,.\,]_{W_q}, \end{aligned}$$

where the functional derivative of $s_1[\,.\,]_{W_q}$ with respect to z_{m+1} is

$$\begin{aligned} &\frac{\delta}{\delta z_{m+1}} s_1[\,.\,]_{W_q} \\ &= s_1\Big[h\, p_D\, G_x^{(|W_q|+1)}(\,.\,) \cdot s_2[L_{z_{m+1}} \mid \cdot] \prod_{z\in W_q} s_2[L_z \mid \cdot]\Big] \\ &= s_1\Big[h\, p_D\, G_x^{(|W_q|+1)}(\,.\,) \prod_{z\in W_q\cup\{z_{m+1}\}} s_2[L_z \mid \cdot]\Big] \\ &= s_1[\,.\,]_{W_q\cup\{z_{m+1}\}} \end{aligned}$$

in which $G_x^{(|W_q|+1)}(\,.\,) = G_x^{(|W_q|+1)}\big(s_2[G_L[g \mid \cdot] \mid \cdot]\big)$. We obtain similar results for the functional derivatives of $C_1^{(|R|)}(\,.\,)$ and $\prod_{W_r\in R} c_1[\,.\,]_{W_r}$ which gives

$$\begin{aligned} &\frac{\delta^2 F}{\delta Z\, \delta z_{m+1}}[g,h] = \\ &\sum_{P\in\mathcal{P}(Z)} \sum_{Q\uplus R=P} G_c^{(|Q|+1)}(\,.\,) \left(\prod_{W_q\in Q\cup\{z_{m+1}\}} s_1[\,.\,]_{W_q}\right) \\ &\qquad \times C_1^{(|R|)}(\,.\,) \left(\prod_{W_r\in R} c_1[\,.\,]_{W_r}\right) \\ &+ \sum_{P\in\mathcal{P}(Z)} \sum_{Q\uplus R=P} G_c^{(|Q|)}(\,.\,) \left(\prod_{W_q\in Q} s_1[\,.\,]_{W_q}\right) \\ &\qquad \times C_1^{(|R|+1)}(\,.\,) \left(\prod_{W_r\in R\cup\{z_{m+1}\}} c_1[\,.\,]_{W_r}\right) \\ &+ \sum_{P\in\mathcal{P}(Z)} \sum_{Q\uplus R=P} C_1^{(|R|)}(\,.\,) \left(\prod_{W_r\in R} c_1[\,.\,]_{W_r}\right) G_c^{(|Q|)}(\,.\,) \\ &\qquad \times \left(\prod_{W_q\in Q} s_1[\,.\,]_{W_q}\right) \sum_{W_q\in Q} \frac{s_1[\,.\,]_{W_q\cup\{z_{m+1}\}}}{s_1[\,.\,]_{W_q}} \\ &+ \sum_{P\in\mathcal{P}(Z)} \sum_{Q\uplus R=P} G_c^{(|Q|)}(\,.\,) \left(\prod_{W_q\in Q} s_1[\,.\,]_{W_q}\right) C_1^{(|R|)}(\,.\,) \\ &\qquad \times \left(\prod_{W_r\in R} c_1[\,.\,]_{W_r}\right) \sum_{W_r\in R} \frac{c_1[\,.\,]_{W_r\cup\{z_{m+1}\}}}{c_1[\,.\,]_{W_r}} \end{aligned} \tag{26}$$

Remark 3. *Note that all partitions of $Z \cup \{z_{m+1}\}$ have the following forms. First, take a partition P of Z and append the cell $\{z_{m+1}\}$ to get $P \cup \{z_{m+1}\}$ of $Z \cup \{z_{m+1}\}$. In relation to the sub-partitions Q and R, where $Q \uplus R = P$ such that Q, $R \subseteq P$, there are two possible partitions of this form, either:*

$$\begin{aligned} P \cup \{z_{m+1}\} = &\ (Q \cup \{z_{m+1}\}) \uplus R, \\ \text{or}\quad P \cup \{z_{m+1}\} = &\ Q \uplus (R \cup \{z_{m+1}\}). \end{aligned}$$

These actions are mathematically represented, respectively, by

$$\begin{aligned} &G_c^{(|Q|+1)}(\,.\,) \left(\prod_{W_q\in Q\cup\{z_{m+1}\}} s_1[\,.\,]_{W_q}\right) \\ \text{and}\quad &C_1^{(|R|+1)}(\,.\,) \left(\prod_{W_r\in R\cup\{z_{m+1}\}} c_1[\,.\,]_{W_r}\right), \end{aligned}$$

from the first two terms in (26).

Alternatively, remove a cell W from P and replace it with the cell $W \cup \{z_{m+1}\}$ to get a new set of partitions $P_{new} = \{\{P \setminus W\} \cup \{W \cup \{z_{m+1}\}\} : \forall\, W \in P\}$. In relation to the sub-partitions Q and R, there are two possible sets of partitions of this form, either:

$$\begin{aligned} & P_{new} = \{(\{Q \setminus W_q\} \cup \{W_q \cup \{z_{m+1}\}\}) \uplus R : \forall\, W_q \in Q\} \\ \textit{or} \quad & P_{new} = \{Q \uplus (\{R \setminus W_r\} \cup \{W_r \cup \{z_{m+1}\}\}) : \forall\, W_r \in R\}. \end{aligned}$$

In other words, the replacement of the cell W for $W \cup \{z_{m+1}\}$ occurs in either the sub-partition Q or R and these actions are mathematically represented, respectively, by the products

$$\begin{aligned} & \left(\prod_{W_q \in Q} s_1[\,.\,]_{W_q}\right) \sum_{W_q \in Q} \frac{s_1[\,.\,]_{W_q \cup \{z_{m+1}\}}}{s_1[\,.\,]_{W_q}} \\ \textit{and} \quad & \left(\prod_{W_r \in R} c_1[\,.\,]_{W_r}\right) \sum_{W_r \in R} \frac{c_1[\,.\,]_{W_r \cup \{z_{m+1}\}}}{c_1[\,.\,]_{W_r}}, \end{aligned}$$

from the last two terms in (26).

Consequently, (26) becomes

$$\begin{aligned} & \frac{\delta F}{\delta (Z \cup \{z_{m+1}\})}[g,h] = \\ & \sum_{P \in \mathcal{P}(Z \cup \{z_{m+1}\})} \sum_{Q \uplus R = P} G_c^{(|Q|)}(\,.\,) \left(\prod_{W_q \in Q} s_1[\,.\,]_{W_q}\right) \\ & \qquad \times\, C_1^{(|R|)}(\,.\,) \left(\prod_{W_r \in R} c_1[\,.\,]_{W_r}\right). \end{aligned}$$

This concludes the inductive step and proves the result in Lemma 1. □

The first moment density, otherwise known as the Probability Hypothesis Density (PHD), of $f_{k+1|k+1}(\mathbb{X}|Z^{(k+1)})$ can be found by taking the functional derivative of $G_{k+1|k+1}[h]$ with respect to h and evaluating at $h = 1$. So from (17) this gives

$$D_{k+1|k+1}(c) = \left(\frac{\delta F}{\delta Z\, \delta(c)}[g,h] \Big/ \frac{\delta F}{\delta Z}[g,h] \right)\Bigg|_{g=0,h=1}. \quad (27)$$

By setting $g = 0$ in (23), we now denote

$$\begin{aligned} G_c^{(|Q|)}(\,.\,) &= G_c^{(|Q|)}\Big(s_1\big[h(q_D + p_D\, p_x(0 \mid \cdot))\big]\Big), \\ s_1[\,.\,]_{W_q} &= s_1\left[h\, p_D\, |W_q|!\, p_x(|W_q| \mid \cdot) \prod_{z \in W_q} s_2[L_z \mid \cdot]\right], \\ c_1[\,.\,]_{W_r} &= c_1\left[C_2^{(|W_r|)}(0 \mid \cdot) \prod_{w \in W_r} c_2(w \mid \cdot)\right], \end{aligned}$$

and re-write the resulting formula so that $\sum_{Q \uplus R = P}$ is replaced by $\sum_{Q \subseteq P}$. Taking the functional derivative with respect to h of this and evaluating at $g = 0$, we have

$$\begin{aligned} & \frac{\delta F}{\delta Z\, \delta c}[g,h]\Bigg|_{g=0} = \\ & \sum_{P \in \mathcal{P}(Z)} \frac{\delta}{\delta c} G_c(\,.\,)\, C_1^{(|P|)}(0) \left(\prod_{W \in P} c_1[\,.\,]_W\right) \\ & + \sum_{P \in \mathcal{P}(Z)} \sum_{Q \subseteq P} \frac{\delta}{\delta c} G_c^{(|Q|)}(\,.\,) \left(\prod_{W_q \in Q} s_1[\,.\,]_{W_q}\right) \\ & \qquad \times\, C_1^{(|P|-|Q|)}(0) \left(\prod_{\bar{W}_q \in P \setminus Q} c_1[\,.\,]_{\bar{W}_q}\right) \\ & + \sum_{P \in \mathcal{P}(Z)} \sum_{Q \subseteq P} G_c^{(|Q|)}(\,.\,) \frac{\delta}{\delta c} \left(\prod_{W_q \in Q} s_1[\,.\,]_{W_q}\right) \\ & \qquad \times\, C_1^{(|P|-|Q|)}(0) \left(\prod_{\bar{W}_q \in P \setminus Q} c_1[\,.\,]_{\bar{W}_q}\right). \quad (28) \end{aligned}$$

The functional derivatives of $G_c^{(|Q|)}(\,.\,)$ and $\prod_{W_q \in Q} s_1[\,.\,]_{W_q}$ with respect to h are found to be

$$\begin{aligned} \frac{\delta}{\delta c} G_c^{(|Q|)}(\,.\,) &= s_1(c)\, G_c^{(|Q|+1)}(\,.\,)\, q_D(c) \\ & \quad +\, s_1(c)\, G_c^{(|Q|+1)}(\,.\,)\, p_x(0 \mid c)\, p_D(c), \end{aligned}$$

likewise for the functional derivative of $G_c(\,.\,)$ and

$$\begin{aligned} & \frac{\delta}{\delta c}\left(\prod_{W_q \in Q} s_1[\,.\,]_{W_q}\right) \\ & = \left(\prod_{W_q \in Q} s_1[\,.\,]_{W_q}\right) \sum_{W_q \in Q} \frac{1}{s_1[\,.\,]_{W_q}} \underbrace{\frac{\delta}{\delta c} s_1[\,.\,]_{W_q}}_{\downarrow}. \\ & s_1(c)\, |W_q|!\, p_x(|W_q| \mid c) \Big(\textstyle\prod_{z \in W_q} s_2[L_z \mid c]\Big)\, p_D(c) \end{aligned}$$

Denoting $L_{W_q}(c) = |W_q|!\, p_x(|W_q| \mid c) \prod_{z \in W_q} s_2[L_z \mid c]$ and re-writing $\sum_{Q \uplus R = P}$ as $\sum_{Q \subseteq P}$, then (28) becomes

$$\begin{aligned} & \frac{\delta F}{\delta Z\, \delta c}[g,h]\Bigg|_{g=0} = \\ & s_1(c) \sum_{P \in \mathcal{P}(Z)} \Bigg\{ G_c^{(1)}(\,.\,)\, C_1^{(|P|)}(0) \left(\prod_{W \in P} c_1[\,.\,]_W\right) \\ & \quad + \sum_{Q \subseteq P} G_c^{(|Q|+1)}(\,.\,) \left(\prod_{W_q \in Q} s_1[h\, p_D\, L_{W_q}]\right) C_1^{(|P|-|Q|)}(0) \\ & \quad \times \left(\prod_{\bar{W}_q \in P \setminus Q} c_1[\,.\,]_{\bar{W}_q}\right) \Bigg\} \Big(q_D(c) + p_x(0 \mid c)\, p_D(c)\Big) \\ & + s_1(c) \sum_{P \in \mathcal{P}(Z)} \Bigg\{ \sum_{Q \subseteq P} G_c^{(|Q|)}(\,.\,) \left(\prod_{W_q \in Q} s_1[h\, p_D\, L_{W_q}]\right) \\ & \quad \times\, C_1^{(|P|-|Q|)}(0) \left(\prod_{\bar{W}_q \in P \setminus Q} c_1[\,.\,]_{\bar{W}_q}\right) \sum_{W_q \in Q} \frac{p_D(c)\, L_{W_q}}{s_1[h\, p_D\, L_{W_q}]} \Bigg\} \quad (29) \end{aligned}$$

After setting $h = 1$ in (29) and substituting into (27) we obtain the desired result for the first moment density of $f_{k+1|k+1}(\mathbb{X}_{k+1} \mid Z^{(k+1)})$ in the case of independent cluster false alarms and missed detections (Theorem 3).

The corresponding result for the *Base Case* can be derived from (27) by removing the summation over sub-partitions Q and the false alarm terms from (29), then setting $p_D = 1$ & $h = 1$, and likewise from (23), setting $p_D = 1$, $g = 0$ & $h = 1$. Substituting the resulting formulae into (27) gives the desired result for the first moment density of $f_{k+1|k+1}(\mathbb{X}_{k+1} \mid Z^{(k+1)})$ in the case of no false alarms or missed detections.

4 Conclusions

In this paper we have proposed an alternative approach for deriving first-order moment densities for the measurement-update equation in the extended object Bayes filter recursion. The approach we've described uses spatial cluster processes to model extended targets rather than specifying a non-standard measurement model that approximates the likelihood function to a Poisson point process. This model further generalises existing extended target frameworks [4, 5, 6] by considering an independent cluster process representation of the extended targets in which the daughter process is completely dependent on its parent.

The results in this paper, presented as they are in their most general form (Theorems 2 & 3), are combinatorially complex. It will be the subject of future research to address this complexity and develop computationally tractable algorithms.

Acknowledgements

The authors are members of the Joint Research Institute in Signal and Imaging Processing within the Edinburgh Research Partnership. Anthony Swain is funded by an EPSRC Industrial CASE Award PhD studentship with Selex Galileo. Daniel Clark is a Royal Academy of Engineering/EPSRC Research Fellow.

References

[1] W. Koch, "On Bayesian tracking of extended objects," *IEEE International Conference, Multisensor Fusion and Integration for Intelligent Systems*, Heidelberg, Germany, September 3–6, 2006, pp. 209–216.

[2] M. Feldmann, and D. Fränken, "Advances on tracking of extended objects and group targets using random matrices." *12th International Conference on Information Fusion*, Seattle, WA, USA, July 6–9, 2009, pp. 1029–1036.

[3] M. Baum, and U. D. Hanebeck, "Extended object tracking based on combined set-theoretic and stochastic fusion," *12th International Conference on Information Fusion*, Seattle, WA, USA, July 6–9, 2009, pp. 1288–1295.

[4] K. Gilholm, S. Godsill, S. Maskell, and D. Salmond, "Poisson models for extended targets and group tracking," *SPIE Conference, Signal and Data Processing of Small Targets*, O. E. Drummond, August 2005, Proc. 5913, pp. 230–241.

[5] R. Mahler, "PHD filters for nonstandard targets, I: Extended targets," *12th International Conference on Information Fusion*, Seattle, WA, USA, July 6–9, 2009, pp. 915–921.

[6] R. Mahler, "CPHD and PHD filters for unknown backgrounds, I: Dynamic data clustering," *SPIE, Sensors and Systems for Space Applications III*, J. L. Cox, and P. Motaghedi, 6 May 2009, Proc. 7330.

[7] D. Daley, and D. Vere-Jones, *An introduction to the theory of point processes*, Springer, 1988.

[8] D. Stoyan, W. Kendall, and J. Mecke, *Stochastic geometry and its applications*, Wiley 2nd ed., New York, 1995.

[9] O. E. Barndorff-Nielsen, W. S. Kendall, and M. N. M. van-Lieshout, *Stochastic geometry: Likelihood and computation*, CRC Press, 1999.

[10] J. E. Moyal, "The general theory of stochastic population processes," *Acta Mathematica*, Vol 108, pp. 1–31, 1962.

[11] G. Mathéron, *Random sets and integral geometry*, J. Wiley, 1975.

[12] J. Neyman, and E. L. Scott, "Statistical approach to problems of Cosmology," *Journal of Royal Statistical Society*, Vol 20, No. 1, pp. 1–29, 1958.

[13] D. R. Cox, P. S. Eagleson, and I. Rodriguez-Iturbe, "Spatial modelling of total storm rainfall," *Proc. of Royal Society of London*, 1986.

[14] A. Brix, and P. J. Diggle, "Spatiotemporal prediction for log-Gaussian Cox processes," *Journal of Royal Statistical Society*, Vol 63, No. 2, pp. 823–841, 2001.

[15] R. Mahler, *Statistical Multisource-Multitarget Information Fusion*, Artech House, Norwood, MA, 2007.

[16] R. Mahler, "Multitarget Bayes filtering via first-order multitarget moments," *IEEE Transactions on Aerospace and Electronic Systems*, Vol 39, No. 4, pp. 1152–2003, 2003.

[17] R. Mahler, "PHD filers of higher order in target number," *IEEE Transactions on Aerospace and Electronic Systems*, Vol 43, No. 4, pp. 1523–1543, 2007.